

The University of Chicago

A WARING PROBLEM

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1938

By

IVAN MORTON NIVEN

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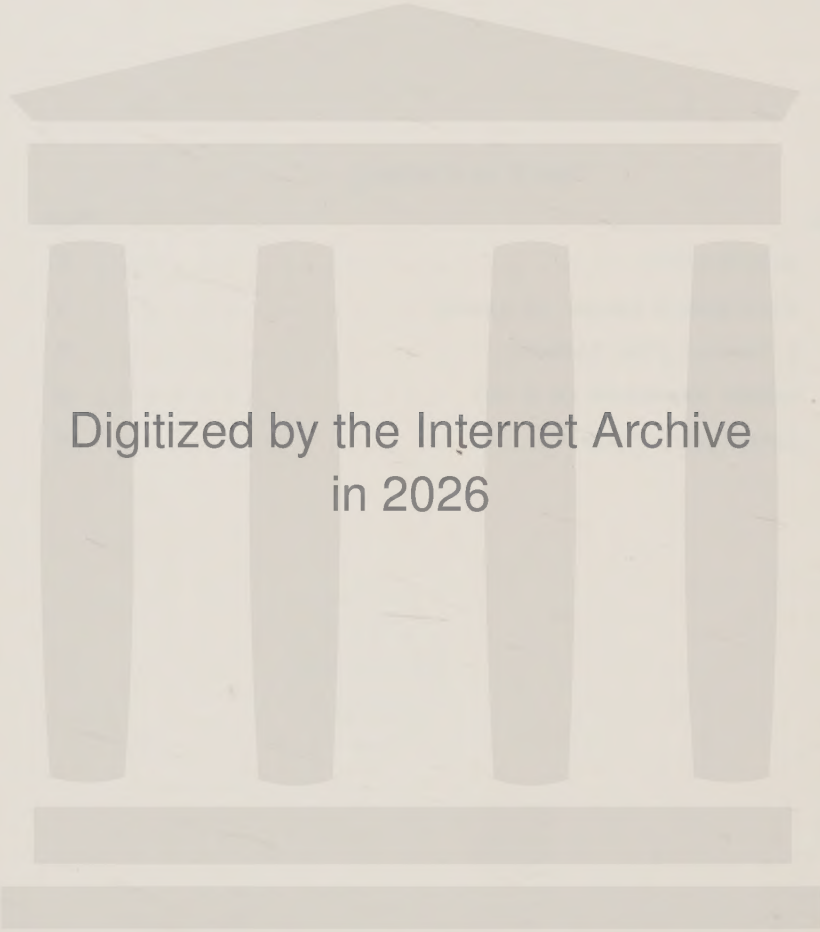
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A WARING PROBLEM

1. Introduction. The original Waring problem was that of evaluating $g(x^n)$, all integers being sums of $g(x^n)$ or fewer positive integral n -th powers, but not all being sums of $g(x^n) - 1$ or fewer. Euler stated that, to express every positive integer as a sum of n -th powers, at least $I = 2^n + q - 2$ terms are necessary, where q is the largest integer less than the quotient of 3^n by 2^n . L. E. Dickson solved this problem (as did S. S. Pillai, independently), showing that $g(x^n) = I$, apart from certain exceptional cases. Using the methods of Dickson, this paper evaluates $g(x^n + 1)$, that is, the minimum number of positive integral n -th powers plus unity required to express all integers. It is proven that $g(x^n + 1) = 2^{n-1} + q - 1 \equiv J$, apart from certain exceptional cases. This J just exceeds $I/2$.

In Sections 2 and 3, asymptotic theory is used to prove the ideal theorem for the cases $11 \leq n \leq 50$. A different approach is used in Section 4 to determine $g(x^n + 1)$ in case $n > 50$. For these latter values of n , there may arise certain exceptions, which are treated in Section 5.

2. Preliminary Lemmas of Ascent. Let $p_x = x^n + 1$. We may write

$$(1) \quad p_3 = p_2 q + r, \quad 0 \leq r \leq 2^n,$$

so that if $[x]$ denotes the greatest integer $\leq x$, we obtain $q = [p_3/p_2]$. By "values" we shall mean any of $0, p_0 = 1, p_1 = 2, p_2 = 2^n + 1, \dots$.

LEMMA 1. All positive integers $\leq p_2q$ are sums of $J = 2^{n-1} + q - 1$ values. But $p_2q - 1$ is not a sum of $J - 1$ values.

For, if $B \leq p_2q - 1$, then $B = p_2x + y$, $0 \leq y \leq 2^n$ and $x < q$. Now $y = [y/2]p_1$ or $y = [y/2]p_1 + p_0$ according as y is even or odd. In either case, y is a sum of $[(y+1)/2]$ values. Hence B is a sum of $x + [(y+1)/2] \leq q - 1 + 2^{n-1} = J$ values.

LEMMA 2. All integers in the interval $(p_2q, p_2q + p_2)$ are sums of E or fewer values, where

$$E = \max(q + 1, q + [r/2], 2^{n-1} + 1 - [r/2]).$$

First, $p_2q, \dots, p_2q + r - 1$ are sums of $q + [r/2]$ values. Second, $p_2q + r = p_3, \dots, p_2q + p_2 - 1 = p_3 - r + p_2 - 1$ are sums of $1 + [(p_2 - r)/2] = 2^{n-1} + 1 - [r/2]$ values. Finally, $p_2q + p_2$ is a sum of $q + 1$ values.

LEMMA 3. Let L, n, z, v be positive integers. If all integers in the interval $(L, L + p_nz)$ are sums of m values, then all in the interval $(L, L + p_nz + p_nv)$ are sums of $m + v$ values.

Let $v = 1$. The proof is needed only for integers x satisfying

$$L + p_nz \leq x \leq L + p_nz + p_n.$$

Since $x - p_n$ lies in $(L, L + p_nz)$, it is a sum of m values, whence x is a sum of $m + 1$ values. An induction on v completes the proof.

LEMMA 4. All in $(p_2q, p_2q + p_2^v)$ are sums of $E + v - 1$ values.

This follows immediately from Lemmas 2 and 3.

Take $v = 1 + q$, so that $p_2^v = p_2q + p_2 > p_3$.

LEMMA 5. All in $(p_2q, p_2q + p_3)$ are sums of $E + q$ values.

This and Lemma 3 with $z = 1$ combine to give

LEMMA 6. All in $(p_2q, p_2q + p_3(1 + v))$ are sums of $E + q + v$ values.

Take $v = [p_4/p_3]$, whence $p_3(1 + v) > p_4$ so that all in $(p_2q, p_2q + p_4)$ are sums of $E + q + [p_4/p_3]$ values. By induction, we obtain

THEOREM 1. All integers in the interval $(p_2q, p_2q + (n + 1)^n)$ are sums of

$$E + q + [p_4/p_3] + \dots + [p_{n+1}/p_n]$$

values and hence sums of

$$(2) \quad E + q + [4^n/3^n] + \dots + [(n + 1)^n/n^n].$$

values.

3. A Theorem from Dickson. Let k_0 be the least integer exceeding $(\log d)/-\log(1 - \nu)$ where $d = n^2(6n - 1)(n - 1 - 2n^2z)$, $z = \nu^3/24$, $\nu = 1/n$. Take $k = 2k_0$, $\log N = 2n^7$ (all logarithms are to the base 10). Then L. E. Dickson¹ has shown that, if $n \geq 9$, every integer $\geq N$ is a sum of $4n - 2 + 3k$ integral n -th powers ≥ 0 . By a more rigorous evaluation of constants employed in the analytic proof of the above, Dickson improves this result so that $\log N = .8n^6$. The same holds true with n -th powers replaced by our integral "values," if n is replaced by $N + 4n - 2 + 3k$, the increase being negligible. Our problem, then, is to extend the interval of Theorem 1 until it includes N . To this end, we employ Lemma 20 from a recent paper by Dickson,² which is stated thus:

¹L. E. Dickson, Proof of the ideal Waring theorem for exponents 7 to 180, American Journal of Mathematics, vol. 58 (1936), p. 525.

²L. E. Dickson, Universal forms $\sum a_1 x_1^n$ and Waring's problem, Acta Arithmetica, vol. 2 (1937), p. 185.

Let g be an integer ≥ 0 , $v = (1 - g/L_0)/n$, $L_0 \geq g + a$, $L_0 v^n \geq a$. L_t is given by $\log L_t = \left(\frac{n}{n-1}\right)^t (\log L_0 + w) - w$, $w = n \log v - \log a$. If all integers between g and L_0 inclusive are represented by a form F , then all between g and L_t inclusive are represented by $F + a(x_1^n + \dots + x_t^n)$.

Let $a = 1$, whence $w = n \log v$. The interval in Theorem 1 includes (g, L_0) where $g = 3^n$ and $L_0 = (n+1)^n$. The inequality $L_0 v^n \geq a$ becomes $v(n+1) \geq 1$ which reduces to

$$(3) \quad \left(\frac{n+1}{3}\right)^{n-1} > 3$$

which is true if $n \geq 4$. For $n > 10$, $v = 1/n$ to seven decimal places. Employing the series for $\log(1+x)$ we obtain

$$\log L_0 - n \log n = n \log(1 + 1/n) > nM(1/n - 1/2n^2)$$

where $M = \log e = .4343$. Then $L_t > N$ if

$$(4) \quad \left(\frac{n}{n-1}\right)^t M(1 - 1/2n) > .8 n^6, \quad n > 10.$$

This is satisfied for $n \leq 50$ by $t = 1196$, and for $n \leq 15$ by $t = 236$. In these cases, the relation

$$(5) \quad E + q + [4^n/3^n] + \dots + [(n+1)^n/n^n] + t \leq J$$

has been verified. This proves

THEOREM 2. For $11 \leq n \leq 50$, every integer is a sum of $J = 2^{n-1} + q - 1$ values.

4. Larger Exponents ($n > 50$). Let $f = [p_4/p_3]$, $g = [p_5/p_4]$ and $s = f + [6g/5]$.

LEMMA 7. If all integers in the interval $(L, L + p_2)$ are sums of m values, then all in $(L, L + p_{n+1})$ are sums of

$$(6) \quad m + q + f + g + [6^n/5^n] + \dots + [(n+1)^n/n^n].$$

This follows from Lemma 3 as in the proof of Theorem 1.

LEMMA 8. Let L be any integer between 2^n and 4^n inclusive.
If $n > 50$ and all integers in $(L, L + p_2)$ are sums of m or fewer
values, then every integer $\geq L$ is a sum of $m + q + s - 2$ values.

Apply the ascent theory of Section 3 with $n > 50$, $g = 3^n$ replaced by $L = 2^n$ and $L = 4^n$ respectively. In the inequality (3), the quantity 3 is then replaced by 2 and 4 respectively; these new inequalities hold for $n > 5$. We must show that the least t satisfying (4), together with (6), does not exceed $m + q + s - 2$. Cancel m, q, f, g and multiply by $(4/5)^n$. It is easily shown that everything on the left decreases as n increases. For, when $n > 50$, (4) may be written

$$(7) \quad t \log \frac{n}{n-1} > 6 \log n + .26968.$$

The coefficient of t is the product of $M = .4343$ by

$$\log_e (1 + \frac{1}{n-1}) > 1/(n-1) - 1/2(n-1)^2 > 1/n.$$

Hence (7) holds if $t > M^{-1}n(.25896 + 6 \log n)$. Using this expression for t , we have that $t(4/5)^n$ decreases when n increases. It is sufficient, then, to verify the initial inequality when $n = 51$ whence $t = 1224$.

LEMMA 9. If $n > 50$ and $s \leq [r/2] \leq 2^{n-1} - q - s$, every
positive integer is a sum of J or fewer values.

The integers $p_2q, \dots, p_2q + r$ are sums of $q + [(r+1)/2]$
 $\leq 2^{n-1} - s + 1$ values. The integers $\geq p_2q + r = p_3$ and $<$
 $p_2(q+1) = p_3 + p_2 - r$ are sums of

$$1 + [(p_2 - r)/2] = 1 + 2^{n-1} - [r/2] \leq 1 + 2^{n-1} - s$$

values. Finally $p_2(q+1)$ is a sum of $q+1$ values. Apply Lemma 8 with $L = p_2q$ and $m = 2^{n-1} + 1 - s$; also use Lemma 1.

LEMMA 10. If $n > 50$ and $3 < [r/2] < s$, every positive integer is a sum of J values.

For $x = 0, 1, 2, \dots$, $u(p_3 - p_2q) - 1 = ur - 1$, $p_2qu + x$ is a sum of $qu + [(x + 1)/2] \leq qu + [ur/2]$ values. If $p_2 > ur$, there exist integers $\geq p_3u$ and $\leq p_2qu + p_2 - 1 = p_3u + p_2 - 1 - ru$ and each is a sum of $u + [(p_2 - ru)/2]$ values. The latter and $qu + [ru/2]$ are $\leq 2^{n-1} - s$ if

$$(8) \quad 2qu + ur \leq 2^n - 2s, \quad ur - 2u \geq 2s.$$

In the first inequality we increase r to $2s$ and decrease it to 8 in the second and obtain

$$(9) \quad s/3 \leq u \leq (2^{n-1} - s)/(q + s).$$

With u in this subinterval, it is clear that our former inequality $p_2 > ur$ is satisfied. There exists an integer u satisfying (8) if the difference between the limits is ≥ 1 . We require

$$3 \cdot 2^{n-1} > qs + s^2 + 6s + 3q,$$

which holds for $n > 50$.

Thus all integers in $(p_2qu, p_2qu + p_2)$ are sums of $2^{n-1} - s$ values. By Lemma 8, every integer $\geq p_2qu$ is a sum of J values.

To complete the proof of Lemma 10, it remains to prove that every positive integer $< 2^n qu$ is a sum of J values. By Lemmas 1, 2, this is true for integers $\leq p_3$. It remains to be proven for the integers between p_3 and p_3U , where U is the least u satisfying (9).

LEMMA 11. All integers in $(p_3w, p_3w + p_2)$ are sums of M values, where M is the maximum of $A = w + 2^{n-1} - [wr/2]$ and $B = qw + 1 + [wr/2]$.

Those $\leq wp_3 + 2^n - wr$ are sums of $w + [(2^n - wr + 1)/2] = w + 2^{n-1} - [wr/2]$ values. The next integer is $p_2wq + p_2$, which, when augmented by 1, 2, ..., $ur - 1$ is expressible as a sum of B values. If $v \leq U - 1$, then $A > B$ since

$$2^{n-1} > qw - w + wr + 1$$

when $w \leq s/3$.

Taking $M = A$ in Lemma 11, we have by Lemma 3 that all integers in $(p_3w, p_3w + p_2(v + 1))$ are sums of $A + v$ values. Now $A + v \leq J$ if $2v \leq 2q + wr - 2w - 2$. Taking maximum v , we have $v + 1 > q$ so that $p_3w + p_2(v + 1) \geq p_3(w + 1)$ for $w = 1, 2, \dots, U - 1$. This completes the proof of Lemma 10.

LEMMA 12. If $n > 50$ and $2^n - 2q - 2s + 2 \leq r \leq 2^n - 2q - 2$, every positive integer is a sum of J values.

For $x = 0, 1, \dots, p_3u - p_2(uq + u - 1) - 1 = p_2 - u(p_2 - r) - 1$, $p_2(uq + u - 1) + x$ is a sum of

$$C = uq + u - 1 + \left[\frac{p_2 - u(p_2 - r)}{2} \right]$$

values. All integers from p_3u to $p_2(uq + u) - 1 = p_3u + u(p_2 - r) - 1$ are sums of

$$D = u + [u(p_2 - r)/2]$$

values. C and D are both $\leq 2^{n-1} - s$ if

$$(10) \quad \frac{2s - 2}{p_2 - r - q - 2} \leq u \leq \frac{2^n - 2s}{2^n + 3 - r}.$$

If we increase r to $2^n - 2q - 2$ on the left and decrease r to $2^n - 2q - 2s + 2$ on the right, we obtain an interval which contains (9) as a subinterval and hence one which still allows u at least one value. The greatest x is positive if $(2^n - r + 1)u < 2^n$, which is obviously true by (10). Lemma 8 applies, whence every integer $\geq p_3u$ is a sum of J values.

We shall prove that the latter is true of the integers in $(p_3^w, p_3^w + p_3)$ for $w = 1, 2, \dots, U - 1$ where U is the least u satisfying (9). Consider the equations

$$(11) \quad \begin{aligned} j\{-p_3 + (q+1)p_2\} &= j(p_2 - r) \quad (j = 1, 2, \dots, w-1), \\ w\{-p_3 + (q+1)p_2\} &= w(p_2 - r). \end{aligned}$$

The difference between any consecutive pair in the first equation is $p_2 - r$. The difference between the second equation value and p_2 is $p_2 - w(p_2 - r)$. This is positive by (10). Adding wp_3 to each of (11), we have that all integers in $(wp_3, wp_3 + w(p_2 - r))$ are sums of $1 + (q+1)(w-1) + [(p_2 - r)/2] = E$ values. Again, the integers in $(wp_3 + w(p_2 - r), wp_3 + p_2)$ are sums of

$$F = w(q+1) + \left\lceil \frac{p_3 - w(p_2 - r)}{2} \right\rceil$$

values. Then $F \geq E$ if

$$(12) \quad p_2 + 2q \geq (w+1)(p_2 - r).$$

The minimum value of r is $2^n - 2q - 2s + 2$ and the maximum of $w+1$ is $s/3$, so we require

$$3(p_2 + 2q) \geq (2q + 2s + 1)s$$

which is true for $n > 50$. This proves that every integer in $(p_3^w, p_3^w + p_2)$ is a sum of F values. Hence all in $(p_3^w, p_3^w + (v+1)p_2)$ are sums of $F + v$ values. Use the largest v for which $F + v \leq J$. The interval ends with $p_3^w + (v+1)p_2$ which is $\geq p_3(w+1)$ if $v+1 > q$. This requires that

$$w(p_2 - r - 2q - 2) > 0$$

which holds since r is at most $2^n - 2q - 2$.

Lemmas 9 to 12 yield

THEOREM 3. If $n > 50$ and $8 \leq r \leq 2^n - 2q - 2$, every positive integer is a sum of J values.

5. Exceptional Cases. The first part of Lemma 10 holds for larger values of r , namely $2^n - 2q - 1 \leq r \leq 2^n - q - 4$. The inequality (10) holds for $r = 2^n - 2q - 1$ and $r = 2^n - q - 4$; further, u has at least one value for each of these values of r ; hence (10) holds and yields a value of u for $2^n - 2q - 1 \leq r \leq 2^n - q - 4$. We have proven

LEMMA 13. If $n > 50$, $2^n - 2q - 1 \leq r \leq 2^n - q - 4$, every positive integer $> 4^n$ is a sum of J values.

We shall next show that r cannot equal any of $2^n - q - 3$, $2^n - q - 2$, ..., $2^n - q + 5$. Let $r = 2^n - q - x$, $x = 0, 1, 2, 3$. Then

$$\begin{aligned} 3^n + 1 &= (2^n + 1)q + 2^n - q - x, \\ 3^n &= 2^n q + 2^n - x - 1. \end{aligned}$$

But L. E. Dickson¹ has shown that if $n \geq 6$, then R lies in the range $5 \leq R \leq 2^n - 5$, where $Q = [3^n/2^n]$.

Again, let $r = 2^n - q + y$, $y = 1, 2, \dots, 5$. Then

$$\begin{aligned} (3^n + 1) &= (2^n + 1)q + 2^n - q + y, \\ 3^n &= 2^n(q + 1) + y - 1. \end{aligned}$$

Since $f = [p_4/p_3]$, we may write

$$(13) \quad p_4 = p_3^f + p_2^h + j, \quad 0 \leq h \leq q, \quad 0 \leq j \leq 2^n.$$

If $h = q$, then $j < r$. Hence if r is restricted as in Lemma 13, namely $2^n - 2q - 1 \leq r \leq 2^n - q - 4$, every positive integer $< 4^n$

¹L. E. Dickson, Solution of Waring's problem, American Journal of Mathematics, vol. 58 (1936), pp. 531, 534.

is a sum of $f + q - 1 + [2^n/2] = 2^{n-1} + q - 1 + f$ values.

We combine all these results in

THEOREM 4. If $n > 50$, $2^n - 2q - 1 \leq r \leq 2^n - q + 5$, every positive integer is a sum of $J + f$ values.

There remain the cases $r \geq 2^n - q + 6$ and $r \leq 7$. Consider the interval $(p_3, p_3 + p_2)$. Each integer therein is a sum of $2^{n-1} + 1$ values. Applying Lemma 8, we have the trivial

THEOREM 5. If $n > 50$ and either $r \geq 2^n - q + 6$ or $r \leq 7$, every integer is a sum of $J + s$ values.

VITA

Ivan Morton Niven was born in Vancouver, Canada, on October 25, 1915. He received his elementary and high school education in the Vancouver city schools. In 1930 he entered the University of British Columbia at Vancouver, graduating in May, 1934, with the degree of Bachelor of Arts. For the next two years he was Assistant in Mathematics at the same institution and was granted the degree of Master of Arts in May, 1936. He was in residence at the University of Chicago as a Fellow in the Department of Mathematics for seven consecutive quarters beginning with the Fall Quarter, 1936. At the University of British Columbia he studied under Professors Buchanan and Nowlan, and at the University of Chicago under Professors Albert, Bartky, Bliss, Compton, Dickson, Graves, Lane, Lunn, and Reid. He wishes to express his indebtedness to all his instructors, and especially to Professor L. E. Dickson, under whose direction this thesis was written.

